

# PERIODIC PERTURBATIONS WITH DELAY OF SEPARATED VARIABLES DIFFERENTIAL EQUATIONS

LUCA BISCONTI AND MARCO SPADINI

**ABSTRACT.** We study the structure of the set of harmonic solutions to perturbed nonautonomous,  $T$ -periodic, separated variables ODEs on manifolds. The perturbing term is allowed to contain a finite delay and to be  $T$ -periodic in time.

## 1. INTRODUCTION

In this paper we study  $T$ -periodic solutions to periodic perturbations of separated variables ODEs on manifolds, allowing the perturbing term to contain a finite delay. Namely, given  $T > 0$ ,  $r \geq 0$  and a boundaryless smooth manifold  $N \subseteq \mathbb{R}^d$ , we consider  $T$ -periodic solutions to equations of the form

$$(1.1) \quad \dot{\zeta}(t) = a(t)\Phi(\zeta(t)) + \lambda\Xi(t, \zeta(t), \zeta(t-r)), \quad \lambda \geq 0,$$

where  $r > 0$  is a finite time lag,  $a: \mathbb{R} \rightarrow \mathbb{R}$  is a continuous  $T$ -periodic function,  $\Phi: N \rightarrow \mathbb{R}^d$  and  $\Xi: \mathbb{R} \times N \times N \rightarrow \mathbb{R}^d$  are given continuous tangent vector fields on  $N$ , in the sense that  $\Phi(\xi)$  belongs to the tangent space  $T_\xi N$ , for any  $\xi \in N$ , and  $\Xi$  is  $T$ -periodic in the first variable and tangent to  $N$  in the second variable, that is

$$\Xi(t, \xi, \eta) = \Xi(t+T, \xi, \eta) \in T_\xi N, \quad \forall (t, \xi, \eta) \in \mathbb{R} \times N \times N.$$

We also assume that the average  $\phi$  of  $a$  is nonzero, i.e.,

$$(1.2) \quad \phi := \frac{1}{T} \int_0^T a(t) dt \neq 0.$$

Clearly,  $a(t)$  can be written as  $\phi + \alpha(t)$  where  $\alpha: \mathbb{R} \rightarrow \mathbb{R}$  is continuous,  $T$ -periodic and with zero average. In this way Equation (1.1) can be obtained by the introduction of a  $T$ -periodic perturbation with null average in the coefficient  $\phi$  in the following equation:

$$\dot{\zeta}(t) = \phi\Phi(\zeta(t)) + \lambda\Xi(t, \zeta(t), \zeta(t-r)), \quad \lambda \geq 0.$$

Our main objective is to provide information on the structure of the set of  $T$ -periodic solutions of (1.1) (recall that  $T > 0$  is given). More precisely, we will give conditions ensuring the existence of a connected set of pairs  $(\lambda, \zeta)$ ,  $\lambda \geq 0$  and  $\zeta$  a  $T$ -periodic solution of (1.1), such that either  $\lambda \neq 0$  or  $\zeta$  is nonconstant, whose closure in an appropriate topological space is not compact and meets the set of pairs formed by constant solutions corresponding to  $\lambda = 0$ .

To pursue our goal we use topological tools as the fixed point index and the degree of tangent vector fields on manifolds (see, e.g. [7]). A deceptively natural approach would be to use a time transformation as in e.g. [16] and then apply directly one of the known results for periodic perturbations of autonomous ODEs

on manifolds as, for instance, [9]. This naive procedure does not work because the time-transformed perturbing term would result in a form difficult to investigate. Our strategy, instead, consists of recasting and combining the arguments of [16] and [9]. To get a general idea of how we proceed, consider the particular case when  $\Phi$  is  $C^1$  (the general case when  $\Phi$  is only continuous boils down to it via an approximation procedure). When  $\alpha \equiv 0$  and the perturbation  $\Xi$  does not depend on the delay, as in [8], our condition is obtained through a formula (see e.g. [7]) relating the degree of the tangent vector field  $\Psi$  to fixed point index of the translation operator at time  $T$ ,  $P_T^\Phi$ , associated to the equation

$$(1.3) \quad \dot{\zeta} = \Phi(\zeta).$$

When  $a$  is constant but the perturbing term in equation (1.1) is allowed to contain a delay, as in [9], one needs to adapt this approach: the operator  $P_T^\Phi$  is replaced with its infinite-dimensional analogous  $Q_T^\Phi$  and the result is obtained through a formula that associates the degree of  $\Phi$  with the fixed point index of  $Q_T^\Phi$ . In the present paper, in order to allow  $a$  to be nonconstant, we revisit the construction in [9] and provide a relation (see Theorem 4.2 below) between the degree of  $\Phi$  with the fixed point index of the *infinite dimensional Poincaré type  $T$ -translation operator*  $Q_T^{a\Phi}$  associated to the separated variables equation  $\dot{\zeta} = a(t)\Phi(\zeta)$ . Namely,  $Q_T^{a\Phi}$  is the operator that associates to any element  $\varphi \in C([-r, 0], N)$  the function given by  $\theta \mapsto \zeta(\varphi(0), \theta + T)$ , with  $\theta \in [-r, 0]$ . Here  $\zeta(p, \cdot)$  denotes the unique solution of the following the Cauchy problem on  $N$ :

$$\dot{\zeta} = a(t)\Phi(\zeta), \quad \zeta(0) = p.$$

Some preliminary results strictly related to this topic can also be found in [1, 9, 16].

In order to illustrate our result, we consider two classes of applications. The first, rather straightforward, to the set of  $T$ -periodic solutions of a particular class of weakly coupled differential equations on manifolds. In the second, we consider delay periodic perturbations to a family of semi-explicit differential-algebraic equations (DAEs). More precisely, we study the structure of the set of  $T$ -periodic solutions to the following problem

$$(1.4) \quad \begin{cases} \dot{x}(t) = a(t)f(x(t), y(t)) + \lambda h(t, x(t), y(t), x(t-r), y(t-r)), & \lambda \geq 0, \\ g(x, y) = 0, \end{cases}$$

where  $r$  is as in (1.1),  $f: U \rightarrow \mathbb{R}^k$ ,  $h: \mathbb{R} \times U \times U \rightarrow \mathbb{R}^k$  and  $g: U \rightarrow \mathbb{R}^s$  are given continuous maps defined on an open connected set  $U \subseteq \mathbb{R}^k \times \mathbb{R}^s \cong \mathbb{R}^d$  and assume that  $h$  is  $T$ -periodic in the variable  $t$ . We also require that  $g \in C^\infty(U, \mathbb{R}^s)$ , with the property that the Jacobian matrix  $\partial_2 g(p, q)$  of  $g$ , with respect to the last  $s$  variables, is invertible for any  $(p, q) \in U$ . Observe that this assumption implies that 0 is a regular value for  $g$ . So,  $g^{-1}(0) \subseteq U$  is a closed  $C^\infty$  submanifold of  $\mathbb{R}^k \times \mathbb{R}^s$  of dimension  $k$ . Throughout the paper we will always denote the manifold  $g^{-1}(0)$  by  $M$ ; in this contest, the points of  $M$  will be written as pairs  $(p, q)$ . It is well-known (see e.g. [12]) that under these hypotheses it is always possible to transform the above DAE into an equivalent ODE of type (1.1) on the differentiable manifold  $M$ . Actually, as a direct consequence of the Implicit Function Theorem,  $M$  can be locally represented as a graph of some map from an open subset of  $\mathbb{R}^k$  to  $\mathbb{R}^s$  and, hence Equation (1.4) can be locally decoupled. However, globally, this might be false or not convenient for our purpose (see, e.g. [3]).

## 2. PRELIMINARIES AND BASIC NOTIONS

In this section, we recall some basic facts and definitions about the function spaces used throughout the paper.

Let  $I \subseteq \mathbb{R}$  be an interval and let  $X \subseteq \mathbb{R}^d$ . Given  $r \in \mathbb{N} \cup \{0\}$ , the set of all  $X$ -valued  $C^r$ -functions defined on  $I$  is denoted by  $C^r(I, X)$ . When  $I = \mathbb{R}$ , we simply write  $C^r(X)$  instead of  $C^r(\mathbb{R}, X)$  and, when  $r = 0$  we simplify the notation writing  $C(I, X)$  in place of  $C^0(I, X)$  and  $C(X)$  instead of  $C^0(X)$ . Let  $T > 0$  be given, by  $C_T(\mathbb{R}^d)$  we mean the Banach space of all the continuous  $T$ -periodic functions  $\zeta: \mathbb{R} \rightarrow \mathbb{R}^d$  whereas  $C_T(X)$  denotes the metric subspace of  $C_T(\mathbb{R}^d)$  consisting of all those  $\zeta \in C_T(\mathbb{R}^d)$  that take values in  $X$ . It not difficult to prove that  $C_T(X)$  is complete if and only if  $X$  is closed in  $\mathbb{R}^d$ .

Let  $N \subseteq \mathbb{R}^d$  be a smooth differentiable manifold, and consider the following diagram of closed embeddings:

$$(2.1) \quad \begin{array}{ccc} [0, \infty) \times N & \longrightarrow & [0, \infty) \times C_T(N) \\ \uparrow & & \uparrow \\ N & \longrightarrow & C_T(N) \end{array}$$

we identify any space in the above diagram with its image. In particular,  $N$  will be regarded as its image in  $C_T(N)$  under the embedding that associates to any  $p \in N$  the function  $\bar{p} \in C_T(N)$  constantly equal to  $p$ . Furthermore, we will regard  $N$  as the slice  $\{0\} \times N \subseteq [0, \infty) \times N$  and, analogously,  $C_T(N)$  as  $\{0\} \times C_T(N)$ . Thus, if  $\Omega$  is a subset of  $[0, \infty) \times C_T(N)$ , then  $\Omega \cap N$  represents the set of points of  $N$  that, regarded as constant functions, belong to  $\Omega$ . Namely, with this convention, we have that

$$(2.2) \quad \Omega \cap N = \{p \in N : (0, \bar{p}) \in \Omega\}.$$

Let  $\Theta: \mathbb{R} \times N \rightarrow \mathbb{R}^d$  be a time-dependent tangent vector field and assume that the Cauchy problem

$$(2.3a) \quad \dot{\zeta} = \Theta(t, \zeta), \quad t \in \mathbb{R},$$

$$(2.3b) \quad \zeta(0) = p,$$

admits unique solution for all  $p \in N$ . Denote by

$$\mathcal{D} = \{(\tau, p) \in \mathbb{R} \times N : \text{the solution of (2.3) is continuable up to } t = \tau\}.$$

A well known argument based on some global continuation properties of the flows (see, e.g. [13]) shows that  $\mathcal{D}$  is open set containing  $\{0\} \times M$ . Let  $P^\Theta: \mathcal{D} \rightarrow N$  be the map that associates to each  $(t, p) \in \mathcal{D}$  the value  $\zeta(t)$  of the maximal solution  $\zeta$  to (2.3), i.e.  $P^\Theta(t, p) = \zeta(t)$ . Here and in the sequel, given  $\tau \in \mathbb{R}$ , we denote by  $P_\tau^\Theta = P^\Theta(\tau, \cdot)$ , the (Poincaré)  $\tau$ -translation operator associated to Equation (2.3a). So that, the domain of  $P_\tau^\Theta$  is an open (possibly empty) set formed by the points  $p \in N$  for which the maximal solution of (2.3a), starting from  $p$  at  $t = 0$ , is defined up to  $\tau$ .

The remark below, borrowed by [16], plays a crucial role in what follows.

**Remark 2.1.** Let  $\Phi: N \rightarrow \mathbb{R}^d$  as in (1.1). Consider the following Cauchy problems

$$(2.4a) \quad \dot{\zeta} = \Phi(\zeta), \quad \zeta(0) = \zeta_0,$$

and

$$(2.4b) \quad \dot{\zeta} = a(t)\Phi(\zeta), \quad \zeta(0) = \zeta_0,$$

with  $\zeta_0 \in N$ . Let  $J$  and  $I$  be the intervals on which are defined the (unique) maximal solutions of (2.4a) and (2.4b) respectively. Suppose also that  $\Phi$  is  $C^1$ , so that the uniqueness of solutions for the above problems is guaranteed. Let  $\xi: J \rightarrow U$  and  $u: I \rightarrow U$  be the maximal solutions of (2.4a) and (2.4b) respectively, with  $I$  and  $J$  the relative maximal intervals of existence.

Let  $t > 0$  be such that  $\int_0^l a(s)ds \in I$  for all  $l \in [0, t]$ , then it follows that

$$\xi(t) = u\left(\int_0^t a(s)ds\right),$$

and hence  $t \in J$ . Conversely, by using a standard maximality argument, one can prove that  $t \in J$  implies  $\int_0^t a(s)ds \in I$ . Assume that the average  $\phi$  of  $a$  is 1. Define the map  $\phi_a: J \rightarrow I$ ,  $t \mapsto \phi_a(t) = \int_0^t a(s)ds$ . Notice that, if  $T \in J$ , then  $\phi_a(T) = T \in I$  and so  $(\xi(T), \sigma(T)) = (x(T), y(T))$ . Observe that  $\phi$  needs not be invertible unless, of course,  $a(t) \neq 0$  for all  $t \in \mathbb{R}$ .

This remark has important consequences in terms of the  $T$ -translation (Poincaré) operators associated to the Cauchy problems (2.4a) and (2.4b). We collect them in the following proposition.

**Proposition 2.2.** *Let  $P_T^{a\Phi}$  and  $P_T^\Phi$  be the  $T$ -translation operators associated to the Cauchy problems (2.4a) and (2.4b), respectively. Then  $P_T^{a\Phi}(\zeta_0)$  is defined if and only if so is  $P_T^\Phi(\zeta_0)$ . In particular, if we assume in addition that the average of  $a$  on  $[0, T]$  is equal to 1, we have  $P_T^{a\Phi}(\zeta_0) = P_T^\Phi(\zeta_0)$ , whenever  $P_T^{a\Phi}$  or  $P_T^\Phi$  is defined.*

*Proof.* Follows immediately from Remark 2.1.  $\square$

The following result is proved in [16, Corollary 2.4]

**Proposition 2.3.** *Let  $\Phi: N \rightarrow \mathbb{R}^k$  be a  $C^1$  tangent vector field, and let  $a: \mathbb{R} \rightarrow \mathbb{R}$  be continuous and  $T$ -periodic with  $\frac{1}{T} \int_0^T a(s)ds = 1$ . Given an open subset  $V$  of  $N$ , if  $\text{ind}(P_T^{a\Phi}, V)$  is well defined, then so is  $\text{ind}(P_T^\Phi, V)$  and*

$$(2.5) \quad \text{ind}(P_T^{a\Phi}, V) = \text{ind}(P_T^\Phi, V) = \deg(-\Phi, V).$$

### 3. THE DEGREE OF TANGENT VECTOR FIELDS ON MANIFOLDS AND SOME RELATED PROPERTIES

We now remind some facts about the notion of degree of tangent vector fields on manifolds. Recall that if  $\Theta: N \rightarrow \mathbb{R}^d$  is a tangent vector field on the differentiable manifold  $N \subseteq \mathbb{R}^d$  which is (Fréchet) differentiable at  $p \in N$  and  $\Theta(p) = 0$ , then the differential  $d_p\Theta: T_p N \rightarrow \mathbb{R}^d$  maps  $T_p N$  into itself (see, e.g., [14]), so that, the determinant  $\det d_p\Theta$  is defined. In the case when  $p$  is a nondegenerate zero (i.e.  $d_p\Theta: T_p N \rightarrow \mathbb{R}^d$  is injective),  $p$  is an isolated zero and  $\det d_p\Theta \neq 0$ . Let  $W$  be an open subset of  $N$  in which we assume  $\Theta$  admissible for the degree, that is we suppose the set  $\Theta^{-1}(0) \cap W$  is compact. Then, it is possible to associate to the pair  $(w, W)$  an integer,  $\deg(w, W)$ , called the degree (or characteristic) of the vector field  $\Theta$  in  $W$  (see e.g. [7, 10]), which, roughly speaking, counts (algebraically) the

zeros of  $\Theta$  in  $W$  in the sense that when the zeros of  $\Theta$  are all non-degenerate, then the set  $\Theta^{-1}(0) \cap W$  is finite and

$$(3.1) \quad \deg(\Theta, W) = \sum_{q \in \Theta^{-1}(0) \cap W} \text{sign det } d_q \Theta.$$

The concept of degree of a tangent vector field is related to the classical one of Brouwer degree (whence its name), but the former notion differs from the latter when dealing with manifolds. In particular, this notion of degree does not need the orientation of the underlying manifolds. However, when  $N = \mathbb{R}^d$ , the degree of a vector field  $\deg(\Theta, W)$  is essentially the well known Brouwer degree of  $\Theta$  on  $W$  with respect to 0. The degree of a tangent vector field satisfies all the classical properties of the Brouwer degree: *Solution, Excision, Additivity, Homotopy Invariance, Normalization* etc. For an ampler exposition of this topic, we refer e.g. to [7, 10, 14].

The Excision property allows the introduction of the notion of *index* of an isolated zero of a tangent vector field. Let  $q \in N$  be an isolated zero of a tangent vector field  $\Theta: N \rightarrow \mathbb{R}^d$ . Obviously,  $\deg(\Theta, V)$  is well defined for any open set  $V \subseteq N$  such that  $V \cap \Theta^{-1}(0) = \{q\}$ . Moreover, by the Excision property, the value of  $\deg(\Theta, V)$  is constant with respect to such  $V$ 's. This common value of  $\deg(\Theta, V)$  is, by definition, the index of  $\Theta$  at  $q$ , and is denoted by  $i(\Theta, q)$ . Using this notation, if  $(\Theta, W)$  is admissible, by the Additivity property we have that if all the zeros in  $W$  of  $\Theta$  are isolated, then

$$\deg(\Theta, W) = \sum_{q \in \Theta^{-1}(0) \cap W} i(\Theta, q).$$

By formula (3.1) we have that if  $q$  is a nondegenerate zero of  $\Theta$ , then  $i(\Theta, q) = \text{sign det } d_q \Theta$ .

**3.1. Tangent vector fields on implicitly defined manifolds.** Let  $\Pi: \mathbb{R} \times N \rightarrow \mathbb{R}^d$  be a continuous time-dependent tangent vector field on the differentiable manifold  $N \subseteq \mathbb{R}^d$ , that is  $\Pi(t, \zeta) \in T_\zeta N$  for each  $(t, \zeta) \in \mathbb{R} \times N$ . Assume that there is a connected open subset  $U$  of  $\mathbb{R}^d$  and a smooth map  $\ell: U \rightarrow \mathbb{R}^s$ ,  $0 < s < d$ , with the property that  $N = \ell^{-1}(0)$ . Suppose also that, up to an orthogonal transformation, one can realize a decomposition  $\mathbb{R}^d = \mathbb{R}^k \times \mathbb{R}^s$  such that the partial derivative,  $\partial_2 \ell(x, y)$ , of  $\ell$  with respect to the second  $s$ -variable is invertible for each  $(x, y) \in U$ .

We will need the following fact.

**Remark 3.1.** *Let  $\Pi$  be as above. Since  $\mathbb{R} \times N$  is a closed subset of the metric space  $\mathbb{R} \times U$ , the well known Tietze's Theorem (see e.g. [4]) implies that there exists a continuous extension  $\widehat{\Pi}: \mathbb{R} \times U \rightarrow \mathbb{R}^d$  of  $\Pi$ .*

This fact can be interpreted as the possibility of “extending” a differential equation on  $N$  to a neighborhood  $U$  of  $N$  in  $\mathbb{R}^d$ . Consider, in fact, the following differential equation on  $N$ :

$$(3.2) \quad \dot{\eta} = \Pi(t, \eta).$$

Let us also consider the following “extended” equation on the neighborhood  $U$  of  $N$  in  $\mathbb{R}^d$ :

$$(3.3) \quad \dot{\eta} = \widehat{\Pi}(t, \eta),$$

where  $\widehat{\Pi}$  is any extension of  $\Pi$  as in Remark 3.1. Observe that the solutions of (3.2) are also solutions of (3.3). Conversely, the solutions of (3.3) that meet  $N$  do actually lie on  $N$  and thus are solutions of (3.2).

Setting  $\eta = (x, y)$ , Equation (3.3) can be conveniently rewritten as follows:

$$(3.4) \quad \begin{cases} \dot{x} = \widehat{\Pi}_1(t, x, y), \\ \dot{y} = \widehat{\Pi}_2(t, x, y), \end{cases}$$

where, according to the above decomposition of  $\mathbb{R}^d$ , for any  $(t, \xi) \in \mathbb{R} \times N$ ,  $\xi = (x, y)$ , we can write that

$$\Pi(t, \xi) = \Pi(t, x, y) = (\Pi_1(t, x, y), \Pi_2(t, x, y)) \in \mathbb{R}^k \times \mathbb{R}^s.$$

As a direct consequence of our assumptions, it follows that

$$(3.5) \quad \Pi_2(t, x, y) = -(\partial_2 \ell(x, y))^{-1} \partial_1 \ell(x, y) \Pi_1(t, x, y).$$

Indeed, the condition  $\Pi(t, \xi) \in T_\xi N$  is equivalent to  $\Pi(t, \xi) \in \ker \ell'(x, y)$ , and one has that, for each  $(t, (x, y)) \in \mathbb{R} \times N$ ,

$$0 = \ell'(x, y) \Pi(t, x, y) = \partial_1 \ell(x, y) \Pi_1(t, x, y) + \partial_2 \ell(x, y) \Pi_2(t, x, y),$$

which implies (3.5). Here  $\ell'(x, y)$  denotes the Fréchet derivative of  $\ell$  at  $(x, y)$ .

Consider now Equation (3.2) on  $N$  again. The simple result below shows that, when the above assumptions on  $g$  hold, (3.2) can be equivalently rewritten as the following differential-algebraic equation:

$$(3.6) \quad \begin{cases} \dot{x} = \widehat{\Pi}_1(t, x, y), \\ \ell(x, y) = 0. \end{cases}$$

Here  $\widehat{\Pi}$  is any continuous extension of  $\Pi$  as in Remark 3.1, and by a *solution* of (3.6) we mean a pair of  $C^1$  functions  $x: J \rightarrow \mathbb{R}^k$  and  $y: J \rightarrow \mathbb{R}^s$ ,  $J$  a nontrivial interval, with the property that  $\dot{x}(t) = \Pi_1(t, x(t), y(t))$  and  $\ell(x(t), y(t)) = 0$  for all  $t \in J$ .

**Lemma 3.2.** *The equation (3.2) is equivalent to the DAE (3.6).*

*Proof.* Let  $x: J \rightarrow \mathbb{R}^k$  and  $y: J \rightarrow \mathbb{R}^s$  be  $C^1$  maps defined on an interval  $J$  with the property that  $t \mapsto \xi(t) = (x(t), y(t))$  is a solution of (3.2). Then, for all  $t \in J$ ,  $\dot{x}(t) = \Pi_1(t, x(t), y(t))$  and, since  $(x(t), y(t)) \in N$ , we have  $\ell(x(t), y(t)) = 0$ .

Conversely, let  $t \mapsto (x(t), y(t))$  be a solution of (3.6). Then, differentiating  $\ell(x(t), y(t)) = 0$  at any  $t \in J$ , one gets

$$\partial_1 \ell(x(t), y(t)) \dot{x}(t) + \partial_2 \ell(x(t), y(t)) \dot{y}(t) = 0.$$

So that

$$\begin{aligned} \dot{y}(t) &= -(\partial_2 \ell(x(t), y(t)))^{-1} \partial_1 \ell(x(t), y(t)) \dot{x}(t) \\ &= -(\partial_2 \ell(x(t), y(t)))^{-1} \partial_1 \ell(x(t), y(t)) \widehat{\Pi}_1(t, x(t), y(t)). \end{aligned}$$

Hence, on account of (3.5),

$$\dot{y}(t) = \widehat{\Pi}_2(t, x(t), y(t)).$$

The assertion follows recalling that the solution meets  $N$ .  $\square$

Let us consider the particular case in which the tangent vector field  $\Pi: \mathbb{R} \times N \rightarrow \mathbb{R}^d$  is defined as

$$\Pi(t, \xi) = a(t)\Phi(\xi), \quad (t, \xi) \in \mathbb{R} \times N,$$

where  $\Phi: N \rightarrow \mathbb{R}^d = \mathbb{R}^k \times \mathbb{R}^s$  is a continuous tangent vector field on  $N = \ell^{-1}(0)$ , with  $\ell: U \rightarrow \mathbb{R}^s$  as above, and  $a: \mathbb{R} \rightarrow \mathbb{R}$  is a continuous map. As in Remark 3.1, Tietze's Theorem implies the existence of a continuous extension  $\widehat{\Phi}: U \rightarrow \mathbb{R}^d$  of  $\Phi$  with  $\widehat{\Phi}|_N \equiv \Phi$ .

Define the map  $F: U \rightarrow \mathbb{R}^k \times \mathbb{R}^s$ , as follows

$$(3.7) \quad F(p, q) := (\widehat{\Phi}_1(p, q), \ell(p, q)),$$

where  $\widehat{\Phi}: U \rightarrow \mathbb{R}^d$  is any extension of  $\Phi$  to  $U$  and  $\widehat{\Phi}_1$  is its first  $\mathbb{R}^k$ -component. The following result (see e.g. [3, Th. 4.1]), allow us to reduce the computation of the degree of the tangent vector field  $\Phi$  on  $N$  to that of the Brouwer degree of the map  $F$  with respect to 0, which is in principle handier. Namely, we have that

**Theorem 3.3.** *Let  $U \subseteq \mathbb{R}^k \times \mathbb{R}^s$  be open and connected, let  $\Psi$  be as above, and  $F: U \rightarrow \mathbb{R}^k \times \mathbb{R}^s$  be given by (3.7). Then, for any  $V \subseteq U$  open, if either  $\deg(\Phi, N \cap V)$  or  $\deg(F, V)$  is well defined, so is the other, and*

$$|\deg(\Phi, N \cap V)| = |\deg(F, V)|.$$

#### 4. POINCARÉ-TYPE TRANSLATION OPERATOR

Let  $N \subseteq \mathbb{R}^d$  be a manifold, and let  $\Phi: N \rightarrow \mathbb{R}^d$  be a tangent vector field on  $N$ . Let  $\Xi: \mathbb{R} \times N \times N \rightarrow \mathbb{R}^d$  be continuous and tangent to  $N$  in the second variable. Given  $T > 0$ , assume also that  $\Xi$  is  $T$ -periodic in  $t$ . Consider the following delay differential equation

$$(4.1) \quad \dot{\zeta}(t) = a(t)\Phi(\zeta(t)) + \lambda\Xi(t, \zeta(t), \zeta(t-r)), \quad \lambda \geq 0,$$

where  $r > 0$  and  $a: \mathbb{R} \rightarrow \mathbb{R}$  is continuous,  $T$ -periodic and with average  $\not\equiv \frac{1}{T} \int_0^T a(t)dt \neq 0$ . We are interested in the  $T$ -periodic solutions of the above equation. Without loss of generality we can assume that  $T \geq r$  (see, e.g. [8]). In fact, for  $n \in \mathbb{N}$ , equation (4.1) and

$$\dot{\zeta}(t) = a(t)\Phi(\zeta(t)) + \lambda\Xi(t, \zeta(t), \zeta(t - (r - nT))), \quad \lambda \geq 0,$$

share the same  $T$ -periodic solutions (although other solutions may be radically different). Thus, if necessary, one can replace  $r$  with  $r - nT$ , where  $n \in \mathbb{N}$  is chosen such that  $0 < r - nT \leq T$ .

Let us now establish some further notation. Given any  $p \in N$ , denote by  $\tilde{p} \in \tilde{N}$  the constant function  $\tilde{p}(t) \equiv p$ ,  $t \in [-T, 0]$ . Moreover, for any  $V \subseteq N$ , and  $W \subseteq \tilde{N}$  we define the sets

$$V^\# := \{\tilde{p} \in \tilde{N} : p \in V\},$$

and

$$W_\# := \{p \in N : \tilde{p} \in W\}.$$

Notice also that, for any given  $V \subseteq N$ , one has  $V^\# \subseteq \tilde{V}$  and  $(\tilde{V})_\# = V$ .

Proceeding as in [8, § 3], we now introduce a Poincaré-type  $T$ -translation operator on an open subset of  $\tilde{N}$ . Here, we assume that  $\Phi$  is  $C^1$ . Let  $Q_T^\Phi$  be the map defined, whenever it makes sense for  $\phi \in \tilde{N}$ , by

$$Q_T^\Phi(\phi)(\theta) = \zeta(\phi(0), T + \theta), \quad \theta \in [-r, 0],$$

where  $\zeta(p, \cdot)$  denotes the unique maximal solution of the Cauchy problem

$$(4.2a) \quad \dot{\zeta}(t) = \Phi(\zeta(t)),$$

$$(4.2b) \quad \zeta(0) = p.$$

Well known properties of differential equations imply that the domain  $\mathcal{D}(Q_T^\Phi)$  of  $Q_T^\Phi$  is an open subset of  $\tilde{N}$ . Moreover, since  $T \geq r$ , the Ascoli-Arzelà Theorem implies that  $Q_T^\Phi$  is a locally compact map (see, e.g. [15]). Observe that, there is a simple relation between  $\mathcal{D}(Q_T^\Phi)$  and  $\mathcal{D}(P_T^\Phi)$ , that is

$$\mathcal{D}(Q_T^\Phi) = \{\varphi \in \tilde{M} : \varphi(0) \in \mathcal{D}(P_T^\Phi)\}.$$

In particular,  $\widetilde{\mathcal{D}(P_T^\Phi)} \subseteq \mathcal{D}(Q_T^\Phi)$ . Notice also that  $P_T^\Phi(p) = Q_T^\Phi(\tilde{p})(0)$  for all  $p \in \mathcal{D}(P_T^\Phi)$ .

Similarly, given  $a: \mathbb{R} \rightarrow \mathbb{R}$  as in (4.1), we define the map  $Q_T^{a\Phi}$  by setting, whenever it makes sense for  $\phi \in \tilde{N}$ ,

$$Q_T^{a\Phi}(\phi)(\theta) = \xi(\phi(0), T + \theta), \quad \theta \in [-r, 0],$$

where  $\xi(p, \cdot)$  is the unique maximal solution of the Cauchy problem

$$(4.3a) \quad \dot{\xi}(t) = a(t)\Phi(\xi(t)),$$

$$(4.3b) \quad \xi(0) = p.$$

Since  $\mathcal{D}(P_T^\Phi) = \mathcal{D}(P_T^{a\Phi})$ , it follows easily that  $\mathcal{D}(Q_T^\Phi) = \mathcal{D}(Q_T^{a\Phi})$ .

It is not difficult to prove that the  $T$ -periodic solutions of (4.2a) are in a one-to-one correspondence with the fixed points of  $Q_T^\Phi$ . Similarly, the  $T$ -periodic solutions to (4.3a) are in a one-to-one correspondence with the fixed points of  $Q_T^{a\Phi}$ . Moreover, if  $\phi = 1$ , Proposition 2.2 imply that the fixed points of  $P_T^\Phi$  coincide with those of  $P_T^{a\Phi}$ . However, even in this case,  $Q_T^\Phi$  might be different from  $Q_T^{a\Phi}$ . We wish to obtain a formula for the fixed point index of admissible pairs  $(Q_T^{a\Phi}, W)$ , with  $W$  open in  $\mathcal{D}(Q_T^{a\Phi})$ . In the case when  $a(t) \equiv 1$ , we have the following result ([9, Theorem 3.2]).

**Theorem 4.1.** *Let  $\Phi$  be as above and let  $W \subseteq \tilde{N}$  be open and such that  $\text{ind}(Q_T^\Phi, W)$  is defined. Then,  $\deg(-\Phi, W_\#)$  is defined as well and*

$$(4.4) \quad \text{ind}(Q_T^\Phi, W) = \deg(-\Phi, W_\#).$$

It is not difficult to see that, for any constant  $c$  and any tangent vector field  $v$ , admissible on an open  $\mathcal{V} \subseteq N$ , one has

$$(4.5) \quad \deg(-cv, \mathcal{V}) = (-\text{sign } c)^{\dim N} \deg(v, \mathcal{V}).$$

Hence, when  $a(t) \equiv \phi$ , Equation (4.4) yields

$$\text{ind}(Q_T^{\phi\Phi}, W) = \deg(-\phi\Phi, W_\#) = (-\text{sign } \phi)^{\dim N} \deg(\Phi, W_\#).$$

We seek to generalize this formula to the case when  $a$  is nonconstant. The first part of our construction follows that of the proof of Theorem 3.2 in [9].

**Theorem 4.2.** *Let  $a$ ,  $\Phi$  and  $T$  be as in (4.1) and let  $Q_T^{a\Phi}$  be as above. Let also  $W \subseteq \tilde{N}$  be open. If the fixed point index  $\text{ind}(Q_T^{a\Phi}, W)$  is defined, then so is  $\deg(\Phi, W_\#)$  and*

$$(4.6) \quad \text{ind}(Q_T^{a\Phi}, W) = (-\text{sign } \phi)^{\dim N} \deg(\Phi, W_\#).$$



In particular, one has that

$$(4.7) \quad \text{ind}(Q_T^{a\Phi}, W) = (-\text{sign } \phi)^{\dim N} \text{ind}(Q_T^\Phi, W).$$

*Proof.* The assumption that  $\text{ind}(Q_T^{a\Phi}, W)$  is defined means that  $W \subseteq \mathcal{D}(Q_T^{a\Phi})$  and that the fixed point set  $\text{Fix}(Q_T^{a\Phi}) \cap W$  is compact. Let us show that  $\deg(\Phi, W_\#)$  is defined too. We need to prove that  $\Phi^{-1}(0) \cap W_\#$  is compact. If  $p \in \Phi^{-1}(0) \cap W_\#$ , then the constant function  $\tilde{p}$  is clearly a fixed point of  $Q_T^{a\Phi}$ . Thus  $\Phi^{-1}(0) \cap W_\#$  is compact since it can be regarded as a closed subset of  $\text{Fix}(Q_T^{a\Phi}) \cap W$ .

We now use the Commutativity Property of the fixed point index in order to obtain a relation between the indices of  $P_T^{a\Phi}$  and  $Q_T^{a\Phi}$ . Define the maps  $h : \mathcal{D}(P_T^{a\Phi}) \rightarrow \widetilde{M}$  and  $k : \widetilde{M} \rightarrow M$  by  $h(p)(\theta) = \xi(p, \theta + T)$  and  $k(\phi) = \phi(0)$ , respectively. Here,  $\xi(p, \cdot)$  indicates the unique maximal solution of the Cauchy problem (4.3). One has that

$$(4.8a) \quad (h \circ k)(\phi)(\theta) = \xi(\phi(0), \theta + T) = Q_T^{a\Phi}(\phi)(\theta), \quad \phi \in \mathcal{D}(Q_T^{a\Phi}), \quad \theta \in [-r, 0],$$

and

$$(4.8b) \quad (k \circ h)(p) = \xi(p, \theta + T)|_{\theta=0} = \xi(p, T) = P_T^{a\Phi}(p), \quad p \in \mathcal{D}(P_T^{a\Phi}).$$

Define  $\gamma = k|_W$ . As a consequence of the Commutativity Property of the fixed point index,  $\text{ind}(h \circ \gamma, \gamma^{-1}(\mathcal{D}(P_T^{a\Phi})))$  is defined if and only if  $\text{ind}(\gamma \circ h, h^{-1}(W))$  is defined as well and, in this case,

$$(4.9) \quad \text{ind}(h \circ \gamma, \gamma^{-1}(\mathcal{D}(P_T^{a\Phi}))) = \text{ind}(\gamma \circ h, h^{-1}(W)).$$

Moreover, since  $W \subseteq \mathcal{D}(Q_T^{a\Phi})$ , then  $\gamma^{-1}(\mathcal{D}(P_T^{a\Phi})) = W$ . Hence, from (4.8), it follows that

$$(4.10a) \quad \text{ind}(Q_T^{a\Phi}, W) = \text{ind}(h \circ \gamma, \gamma^{-1}(\mathcal{D}(P_T^{a\Phi}))),$$

and

$$(4.10b) \quad \text{ind}(P_T^{a\Phi}, h^{-1}(W)) = \text{ind}(\gamma \circ h, h^{-1}(W)).$$

Recall that, according to remark 2.1,  $\mathcal{D}(P_T^\Phi) = \mathcal{D}(P_T^{a\Phi})$  so that  $h^{-1}(W) \subseteq \mathcal{D}(P_T^\Phi)$ . Then by (4.9) and (4.10) we get

$$(4.11) \quad \text{ind}(Q_T^{a\Phi}, W) = \text{ind}(P_T^{a\Phi}, h^{-1}(W)).$$

Assume now in addition that the average  $\phi$  of  $a$  is equal to 1. Proposition 2.3 yields

$$(4.12) \quad \text{ind}(P_T^{a\Phi}, h^{-1}(W)) = \text{ind}(P^\Phi, h^{-1}(W)) = \deg(-\Phi, h^{-1}(W)).$$

By the definition of  $h$ , one has that  $\Phi^{-1}(0) \cap W_\# = \Phi^{-1}(0) \cap h^{-1}(W)$ . In fact, all the constant solutions of (4.3a) lie in  $W_\#$ . Then, from the Excision Property of the degree of a tangent vector field, we get

$$(4.13) \quad \deg(-\Phi, h^{-1}(W)) = \deg(-\Phi, W_\#).$$

Therefore, we get (4.6) by (4.11), (4.12) and (4.13).

Let us now remove the additional assumption on  $a$ . Let us put  $a_0(t) = a(t)/\phi$  for all  $t \in \mathbb{R}$  and  $\Phi_a(p) = \phi \Phi(p)$  for all  $p \in N$ . We rewrite equation (4.3) as follows:

$$\dot{\xi}(t) = a_0(t)\Phi_a(\xi(t)),$$

and observe that  $Q_T^{a_0\Phi_a} = Q_T^{a\Phi}$ . Since the average of  $a_0$  over  $[0, T]$  is equal to 1 we get, using the first part of the proof,

$$(4.14) \quad \text{ind}(Q_T^{a\Phi}, W) = \deg(-\phi\Phi, W_{\#}).$$

Since from (4.5) we have

$$\deg(-\phi\Phi, W_{\#}) = (-\text{sign } \phi)^{\dim N} \deg(\Phi, W_{\#}),$$

the assertion follows from (4.14).  $\square$

## 5. BRANCHES OF STARTING PAIRS TO (1.1)

Any pair  $(\lambda, \varphi) \in [0, \infty) \times \tilde{N}$  is said to be a *starting pair* for (1.1) if the following initial value problem has a  $T$ -periodic solution:

$$(5.1) \quad \begin{cases} \dot{\zeta}(t) = a(t)\Phi(\zeta(t)) + \lambda\Xi(t, \zeta(t), \zeta(t-r)), & t > 0, \\ \zeta(t) = \varphi(t), & t \in [-r, 0]. \end{cases}$$

A pair of the type  $(0, \tilde{p})$  with  $\Phi(p) = 0$  is clearly a starting pair and will be called a *trivial starting pair*. The set of all starting pairs for (1.1) will be denoted by  $S$ . For the reminder of this section we assume that  $\Phi$  and  $\Xi$  are  $C^1$ , so that (5.1) admits a unique solution that we denote by  $\xi^\lambda(\varphi, \cdot)$ . By known continuous dependence properties of delay differential equations the set  $\mathcal{V} \subseteq [0, \infty) \times \tilde{N}$  given by

$$\mathcal{V} := \{(\lambda, \varphi) \in [0, \infty) \times \tilde{N} : \xi^\lambda(\varphi, \cdot) \text{ is defined on } [0, T]\}$$

is open. Clearly  $\mathcal{V}$  contains the set  $S$  of all starting pairs for (1.1). Observe that  $S$  is closed in  $\mathcal{V}$ , even if it may be not so in  $[0, +\infty) \times \tilde{N}$ . Moreover, by the Ascoli-Arzelà Theorem it follows that  $S$  is locally compact.

It is convenient to introduce the following notation for the “slices” of product spaces. Let  $Y$  be a set. Given  $X \subseteq [0, \infty) \times Y$ , we put  $X_\lambda = \{\varphi \in Y : (\lambda, \varphi) \in X\}$  for each  $\lambda \geq 0$ .

The following is our main result concerning starting pairs:

**Theorem 5.1.** *Assume that  $\Phi, \Xi, S$  are as above and let  $a: \mathbb{R} \rightarrow \mathbb{R}$  be continuous and  $T$ -periodic such that its average on a period is nonzero. Let  $W \subseteq [0, \infty) \times \tilde{N}$  be open. If  $\deg(\Phi, (W_0)_{\#})$  is defined and nonzero, then the set*

$$(S \cap W) \setminus \{(0, \tilde{p}) \in W : \Phi(p) = 0\}$$

*of nontrivial starting pairs in  $W$ , admits a connected subset whose closure in  $S \cap W$  meets  $\{(0, \tilde{p}) \in W : \Phi(p) = 0\}$  and is not compact.*

The proof of Theorem 5.1 is based on the following global connectivity result of [6] and follows closely that of Proposition 4.1 of [9]. We adapt it here for the sake of completeness.

**Lemma 5.2.** *Let  $Y$  be a locally compact metric space and let  $Z$  be a compact subset of  $Y$ . Assume that any compact subset of  $Y$  containing  $Z$  has nonempty boundary. Then  $Y \setminus Z$  contains a connected set whose closure (in  $Y$ ) intersects  $Z$  and is not compact.*

*Proof of Theorem 5.1.* Consider the open set  $U = W \cap \mathcal{V}$ . Since  $\Phi^{-1}(0) \cap (U_0)_{\#} = \Phi^{-1}(0) \cap (W_0)_{\#}$ , and  $S \cap U = S \cap W$ , we need to prove that the set of nontrivial starting pairs in  $U$  admits a connected subset whose closure in  $S \cap U$  meets the set

$\{(0, \tilde{p}) \in U : \Phi(p) = 0\}$  and is not compact. We deduce this fact from Lemma 5.2 applied to the pair

$$(Y, Z) = \left( S \cap U, \{(0, \tilde{p}) \in U : p \in \Phi^{-1}(0)\} \right).$$

In fact, if a connected set is as in Lemma 5.2 then its closure satisfies all the requirements.

Since  $U$  is open and  $S$  is locally compact  $S \cap U$  is locally compact too. Moreover, the assumption that  $\deg(\Phi, (W_0)_\#)$  is defined means that the set

$$\{p \in (W_0)_\# : \Phi(p) = 0\} = \{p \in (U_0)_\# : \Phi(p) = 0\}$$

is compact. Thus the homeomorphic set  $\{(0, \tilde{p}) \in U : \Phi(p) = 0\}$  is compact as well. Let us now prove that there exists no compact subset of  $S \cap U$  containing  $Z$  and with empty boundary in  $S \cap U$ .

Assume by contradiction that such a set, call it  $C$ , exists. Then  $C$  is relatively open in  $S \cap U$  and  $(S \cap U) \setminus C$  is closed in  $S \cap U$ . Hence, the distance  $\delta = \text{dist}(C, (S \cap U) \setminus C)$  between the compact set  $C$  and  $(S \cap U) \setminus C$  is nonzero. Define, for each  $\lambda \geq 0$ , the map  $\mathcal{Q}_\lambda : \mathcal{V}_\lambda \rightarrow \widetilde{M}$  given by

$$\mathcal{Q}_\lambda(\varphi)(\theta) = \xi^\lambda(\varphi, \theta + T), \quad \theta \in [-r, 0].$$

Notice that  $\mathcal{Q}_0$  coincides with the map  $Q_T^{a\Phi}$  defined in the previous section. In fact, if  $\zeta(p, \cdot)$  is the unique solution of the Cauchy problem (4.3), then we have  $\xi^0(\varphi(0), \cdot) = \zeta(\varphi(0), \cdot)$ . Consider the set

$$A = \{(\lambda, \varphi) \in U : \text{dist}((\lambda, \varphi), C) < \delta/2\},$$

which, clearly, does not meet  $(S \cap U) \setminus C$ . The compactness of  $S \cap U \cap A = C$  imply that for some  $\lambda_* > 0$  one has  $(\{\lambda_*\} \times A_{\lambda_*}) \cap S \cap U = \emptyset$ . So, since the set  $S \cap U \cap A$  coincides with  $\{(\lambda, \varphi) \in A : \mathcal{Q}_\lambda(\varphi) = \varphi\}$ , the Generalized Homotopy Invariance Property of the fixed point index imply that

$$0 = \text{ind}(\mathcal{Q}_{\lambda_*}, A_{\lambda_*}) = \text{ind}(\mathcal{Q}_0, A_0),$$

Thus, as  $Q_T^{a\Phi} = \mathcal{Q}_0$ , Theorem 4.2 and the Excision Property of the degree yield

$$0 = \text{ind}(\mathcal{Q}_0, A_0) = \text{ind}(Q_T^{a\Phi}, A_0) = |\deg(\Phi, (A_0)_\#)| = |\deg(\Phi, (W_0)_\#)|,$$

against the assumption.  $\square$

## 6. BRANCHES OF $T$ -PERIODIC PAIRS TO (1.1)

In this section we focus on the  $T$ -periodic solutions to (1.1). In fact, we study the topological structure of the set of pairs  $(\lambda, x) \in [0, \infty) \times C_T(N)$  where  $x$  is a solution of this equation. Our result extends that of [9, §5]

Recall that  $\Phi : N \rightarrow \mathbb{R}^d$ ,  $\Xi : \mathbb{R} \times N \times N \rightarrow \mathbb{R}^d$  and  $a : \mathbb{R} \rightarrow \mathbb{R}$  are continuous with  $\Phi$  and  $\Xi$  tangent to  $N$  in the sense specified in the Introduction. We also assume that  $a$  and  $\Xi$  are  $T$ -periodic in  $t$  and  $a$  has nonzero average on a period.

A pair  $(\lambda, \zeta) \in [0, \infty) \times C_T(N)$ , where  $\zeta$  is a  $T$ -periodic solution of (1.1), is called a  $T$ -periodic pair. Those  $T$ -periodic pairs that are of the particular form  $(0, \bar{p})$  are said to be *trivial*. Notice that, since  $a$  is not identically zero,  $(0, \bar{p}) \in [0, \infty) \times C_T(N)$  is a trivial  $T$ -periodic pair if and only if  $\Phi(p) = 0$ . We point out that if  $\zeta$  is a nonconstant  $T$ -periodic solution of the unperturbed equation  $\dot{\zeta} = a(t)\Phi(\zeta)$ , then  $(0, \zeta)$  is a nontrivial  $T$ -periodic pair.

The following is our main result. Its proof follows closely that of [9, Thm. 5.1] (which, in turn, is inspired to [6]). In fact, the only remarkable difference is related to the use of Theorem 5.1. For the sake of completeness, however, we restate the argument here in a slightly more schematic form.

**Theorem 6.1.** *Let  $a$ ,  $\Phi$  and  $\Xi$  as above. Let  $\Omega \subseteq [0, \infty) \times C_T(N)$  be open and such that  $\deg(\Phi, \Omega \cap N)$  is defined and nonzero. Then  $\Omega$  contains a connected set of nontrivial  $T$ -periodic pairs for (1.1) whose closure in  $\Omega$  meets the set  $\{(0, \bar{p}) \in \Omega : \Phi(p) = 0\}$  and is not compact. In particular, the set of  $T$ -periodic pairs for (1.1) contains a connected component that meets  $\{(0, \bar{p}) \in \Omega : \Phi(p) = 0\}$  and whose intersection with  $\Omega$  is not compact.*

The following lemma takes care of a special case.

**Lemma 6.2.** *Let  $a$ ,  $\Phi$ ,  $\Xi$  and  $\Omega$  be as in Theorem 6.1. Assume in addition that  $\Phi$  and  $\Xi$  are  $C^1$ . Then  $\Omega$  contains a connected set of nontrivial  $T$ -periodic pairs for (1.1) whose closure in  $\Omega$  meets the set  $\{(0, \bar{p}) \in \Omega : \Phi(p) = 0\}$  and is not compact.*

*Proof.* Denote by  $X$  the set of  $T$ -periodic pairs of (1.1) and by  $S$  the set of starting pairs of the same equation. Define the map  $h : X \rightarrow S$  by  $h(\lambda, \zeta) = (\lambda, \zeta|_{[-r, 0]})$  and observe that  $h$  is continuous and onto. Since  $\Phi$  and  $\Xi$  are  $C^1$ , then  $h$  is also one to one. Observe that, the trivial solution pairs correspond to trivial starting points under this homeomorphism. Moreover, by continuous dependence on data,  $h^{-1} : S \rightarrow X$  is continuous as well. Consider the set

$$S_\Omega = \{(\lambda, \varphi) \in S : \text{the solution of (1.1) is contained in } \Omega\}$$

so that  $X \cap \Omega$  and  $S_\Omega$  correspond under the transformation  $h : X \rightarrow S$ . Thus,  $S_\Omega$  is an open subset of  $S$ . Consequently, we can find an open subset  $W$  of  $[0, \infty) \times \tilde{N}$  such that  $S \cap W = S_\Omega$ . This implies the following chain of equalities:

$$\begin{aligned} \{p \in (W_0)_\# : \Phi(p) = 0\} &= \{p \in N : (0, \tilde{p}) \in W, \Phi(p) = 0\} = \\ &= \{p \in N : (0, \bar{p}) \in \Omega, \Phi(p) = 0\} = \{p \in \Omega \cap N : \Phi(p) = 0\}. \end{aligned}$$

Hence, by excision, it follows that  $\deg(\Phi, (W_0)_\#) = \deg(\Phi, \Omega \cap N) \neq 0$ . By appealing to Theorem 5.1, we find that there exists a connected set

$$\Sigma \subseteq (S \cap W) \setminus \{(0, \tilde{p}) \in W : \Phi(p) = 0\}$$

whose closure in  $S \cap W$  meets  $\{(0, \tilde{p}) \in W : \Phi(p) = 0\}$  and is not compact.

The set  $\Gamma = h^{-1}(\Sigma) \subseteq X \cap \Omega$  is a connected set of nontrivial  $T$ -periodic pairs whose closure in  $X \cap \Omega$  meets  $\{(0, \bar{p}) \in \Omega : \Phi(p) = 0\}$  and is not compact. Since  $X$  is closed in  $[0, \infty) \times C_T(N)$ , the closures of  $\Gamma$  in  $X \cap \Omega$  and in  $\Omega$  coincide. Therefore  $\Gamma$  satisfies the requirements.  $\square$

The proof of Theorem 6.1 can be now performed through an approximation procedure.

*Proof of Theorem 6.1.* As in the last part of the proof of Lemma 6.2, it is enough to show the existence of a connected set  $\Gamma$  of nontrivial  $T$ -periodic pairs whose closure in  $X \cap \Omega$  meets  $\{(0, \bar{p}) \in \Omega : \Phi(p) = 0\}$  and is not compact.

Observe that the closed subset  $X$  of  $[0, \infty) \times C_T(N)$  is locally compact because of Ascoli-Arzelà Theorem. It is convenient to introduce the following subset of  $X$ :

$$\Upsilon = \{(0, \bar{p}) \in [0, \infty) \times C_T(N) : \Phi(p) = 0\}.$$

Take  $Y = X \cap \Omega$  and  $Z = \Upsilon \cap \Omega$  and notice that  $Y$  is locally compact as an open subset of  $X$ . Moreover,  $Z$  is a compact subset of  $Y$  (recall that, by assumption,  $\deg(\Psi, N \cap \Omega)$  is defined). Since  $Y$  is closed in  $\Omega$ , we only have to prove that the pair  $(Y, Z)$  satisfies the hypothesis of Lemma 5.2. Assume the contrary. Thus, we can find a relatively open compact subset  $C$  of  $Y$  containing  $Z$ . Similarly to the proof of Proposition 5.1, given  $0 < \rho < \text{dist}(C, Y \setminus C)$ , we consider the set  $A^\rho$  of all pairs  $(\lambda, \varphi) \in \Omega$  whose distance from  $C$  is smaller than  $\rho$ . Thus,  $A^\rho \cap Y = C$  and  $\partial A^\rho \cap Y = \emptyset$ . We can also assume that the closure  $\overline{A^\rho}$  of  $A^\rho$  in  $[0, \infty) \times C_T(N)$  is contained in  $\Omega$ . Since  $C$  is compact and  $[0, \infty) \times N$  is locally compact, we can take  $A^\rho$  in such a way that the set

$$\{(\lambda, x(t), x(t-r)) \in [0, \infty) \times N \times N : (\lambda, x) \in A^\rho, t \in [0, T]\}$$

is contained in a compact subset of  $[0, \infty) \times N \times N$ . This implies that  $A^\rho$  is bounded with complete closure and  $A^\rho \cap N$  is a relatively compact subset of  $\Omega \cap N$ . In particular  $\Phi$  is nonzero on the boundary of  $A^\rho \cap N$  (relative to  $N$ ). Known approximation results on manifolds yield sequences  $\{\Phi_i\}_{i \in \mathbb{N}}$  and  $\{\Xi_i\}_{i \in \mathbb{N}}$  of  $C^1$  maps uniformly approximating  $\Phi$  and  $\Xi$ , respectively, and such that for all  $i \in \mathbb{N}$  we have

- (a)  $\Phi_i(p) \in T_p N$  for all  $p \in N$ ;
- (b)  $\Xi_i(t, p, q) \in T_p N$  for all  $(t, p, q) \in \mathbb{R} \times N \times N$ ;
- (c)  $\Xi_i$  is  $T$ -periodic in the first variable.

Thus, for  $i \in \mathbb{N}$  large enough, we get

$$\deg(\Phi_i, A^\rho \cap N) = \deg(\Phi, A^\rho \cap N).$$

Furthermore, by excision,

$$\deg(\Phi, A^\rho \cap N) = \deg(\Phi, \Omega \cap N) \neq 0.$$

Therefore, given  $i$  large enough, Lemma 6.2 can be applied to the equation

$$(6.1) \quad \dot{x}(t) = a(t)\Phi_i(x(t)) + \lambda\Xi_i(t, x(t), x(t-r)).$$

Let  $X_i$  denote the set of  $T$ -periodic pairs of (6.1) and put

$$\Upsilon_i = \{(0, \overline{p}) \in [0, \infty) \times C_T(N) : \Phi_i(p) = 0\}.$$

Lemma 6.2 yields a connected subset  $\Gamma_i$  of  $A^\rho$  whose closure in  $A^\rho$  meets  $\Upsilon_i \cap A^\rho$  and is not compact. Let us denote by  $\overline{\Gamma}_i$  and  $\overline{A^\rho}$  the closures in  $[0, \infty) \times C_T(N)$  of  $\Gamma_i$  and  $A^\rho$ , respectively.

We claim that, for  $i$  large enough,  $\overline{\Gamma}_i \cap \partial A^\rho \neq \emptyset$ . Thus,  $X_i$  being closed, we get  $\overline{\Gamma}_i \subseteq X_i$  and this implies the existence of a  $T$ -periodic pair  $(\lambda_i, x_i) \in \partial A^\rho$  of (6.1).

Notice that proving the claim boils down to showing that  $\overline{\Gamma}_i$  is compact. In fact, if the latter assertion is true and we assume  $\overline{\Gamma}_i \cap \partial A^\rho = \emptyset$ , then we get  $\overline{\Gamma}_i \subseteq A^\rho$  so that the closure of  $\Gamma_i$  in  $A^\rho$  coincides with the compact set  $\overline{\Gamma}_i$ . But this is a contradiction. The compactness of  $\overline{\Gamma}_i$  for  $i$  large enough follows from the completeness of  $\overline{A^\rho}$  and the fact that, by the Ascoli-Arzelà Theorem,  $\overline{\Gamma}_i$  is totally bounded, when  $i$  is sufficiently large. Hence the claim is proved.

Again by Ascoli-Arzelà Theorem, we may assume that, as  $i \rightarrow \infty$ ,  $x_i \rightarrow x_0$  in  $C_T(N)$  and  $\lambda_i \rightarrow \lambda_0$  with  $(\lambda_0, x_0) \in \partial A^\rho$ . Passing to the limit in equation (6.1), it is not difficult to show that  $(\lambda_0, x_0)$  is a  $T$ -periodic pair for (1.1) in  $\partial A^\rho$ . This contradicts the assumption  $\partial A^\rho \cap Y = \emptyset$  and proves the first part of the assertion.

Let us prove the last part of the thesis. Consider the connected component  $\Xi$  of  $X$  that contains the connected set  $\Gamma$  of the first part of the assertion. We shall now show that  $\Xi$  has the required properties. Clearly,  $\Xi$  meets the set  $\{(0, \bar{p}) \in \Omega : \Phi(p) = 0\}$  because the closure  $\bar{\Gamma}^\Omega$  of  $\Gamma$  in  $\Omega$  does. Moreover,  $\Xi \cap \Omega$  cannot be compact, since  $\Xi \cap \Omega$ , as a closed subset of  $\Omega$ , contains  $\bar{\Gamma}^\Omega$ , and  $\bar{\Gamma}^\Omega$  is not compact. This completes the proof.  $\square$

## 7. APPLICATIONS AND EXAMPLES

The purpose of this section is to illustrate the techniques developed and the results obtained in the foregoing ones. In order to do so, we will examine two classes of separated variables perturbed differential equations. Namely, perturbed decoupled systems and differential-algebraic equations of a certain form.

**7.1. Weakly coupled equations.** Here, we consider delay periodic perturbations of a particular family of ordinary differential equations on product manifolds. Namely, if  $N_1 \subseteq \mathbb{R}^{n_1}$  and  $N_2 \subseteq \mathbb{R}^{n_2}$  are boundaryless smooth manifolds, we consider the following differential equation on  $N = N_1 \times N_2$ :

$$(7.1) \quad \begin{cases} \dot{x}_1(t) = a_1(t)\Phi_1(x_1(t)) + \lambda\Xi_1(t, x(t), x(t-r)), \\ \dot{x}_2(t) = a_2(t)\Phi_2(x_2(t)) + \lambda\Xi_2(t, x(t), x(t-r)), \end{cases}$$

where  $\Phi_1: N_1 \rightarrow \mathbb{R}^{n_1}$ ,  $\Phi_2: N_2 \rightarrow \mathbb{R}^{n_2}$ , are (continuous) tangent vector fields,  $\Xi_1: \mathbb{R} \times N \times N \rightarrow \mathbb{R}^{n_1}$ ,  $\Xi_2: \mathbb{R} \times N \times N \rightarrow \mathbb{R}^{n_2}$  and the maps  $a_i: \mathbb{R} \rightarrow \mathbb{R}$ ,  $i = 1, 2$ , are continuous,  $T$ -periodic in  $t$ . We also assume that, for  $i = 1, 2$ , the average of  $\phi_i$  of  $a_i$  is nonzero and that  $\Xi_i$  is tangent to  $N_i$  in the second variable.

Clearly, when  $\lambda = 0$ , the resulting unperturbed equations are completely decoupled. In essence, the perturbation provides the (only) coupling in (7.1).

We point out that equations of this form arise naturally from models for a real system. Indeed, the equations (7.1) are inspired to a model (see, e.g. [5, Chapter 2]) describing the interactions of two species that share the same habitat and feed on the same resource. Namely, if  $x_1(t)$  and  $x_2(t)$  are the densities of such a competing species at time  $t$ , the model is as follows:

$$(7.2) \quad \begin{cases} \dot{x}_1 = a_1x_1 + x_1(a_{12}x_2 - a_{11}x_1), \\ \dot{x}_2 = a_2x_2 + x_2(a_{22}x_2 - a_{21}x_1), \end{cases}$$

where  $a_i, a_{j,k}, i, j, k = 1, 2$ , are positive quantities and  $a_1, a_2$  represent the “intrinsic” growth rates of the two species,  $a_{11}, a_{22}$  give the strength of the intraspecific competition and  $a_{12}, a_{21}$  the strength of the interspecific competition. It may be reasonable (see e.g. [11]) to assume that the intrinsic growth rates undergo periodic fluctuations. We can model this by letting  $a_i, i = 1, 2$ , be periodic functions. If we also suppose that the resource on which  $x$  and  $y$  feed takes time  $r$  to recover, we are led to the following modification of (7.2):

$$\begin{cases} \dot{x}_1(t) = a_1(t)x_1(t) + x_1(t)(a_{12}x_2(t-r) - a_{11}x_1(t-r)), \\ \dot{x}_2(t) = a_2(t)x_2(t) + x_2(t)(a_{22}x_2(t-r) - a_{21}x_1(t-r)). \end{cases}$$

Let us point out that more general examples in the same direction are considered in a number of models for the dynamics of animals populations (see, e.g. [2, 11]).

The system (7.1), where a parameter  $\lambda \geq 0$  has been added can be thought as a far-reaching generalization of the above system. However, our little digression above is only intended as justification of our interest in Equation (7.1). In fact,

in this section, we consider solutions of this equation regardless of any possible ecological meaning.

Even if Theorem 6.1 cannot be applied directly to (7.1), we can use the same strategy. We only sketch the argument. As in Remark 2.1, assume that, for  $i = 1, 2$ ,  $\Phi_i$  are  $C^1$  so that uniqueness of the solutions of the following initial value problems on  $N = N_1 \times N_2 = N$  hold:

$$(7.3a) \quad \dot{x}_1 = \Phi_1(x_1), \quad \dot{x}_2 = \Phi_2(x_2), \quad x(0) = \xi_0,$$

and

$$(7.3b) \quad \dot{x}_1 = a_1(t)\Phi_1(x_1), \quad \dot{x}_2 = a_2(t)\Phi_2(x_2), \quad x(0) = \xi_0,$$

and let  $\xi: J \rightarrow N$  and  $u: I \rightarrow N$  be the maximal solutions of (7.3a) and (7.3b), respectively, with  $I$  and  $J$  the relative maximal intervals of existence. Let  $t > 0$  be such that  $\int_0^l a_i(s)ds \in I$ ,  $i = 1, 2$ , for all  $l \in [0, t]$ , then it follows that

$$\xi(t) = (\xi_1(t), \xi_2(t)) = \left( u_1 \left( \int_0^t a_1(s)ds \right), u_2 \left( \int_0^t a_2(s)ds \right) \right),$$

and hence  $t \in J$ . Conversely, by a maximality argument, it can be shown that  $T \in J$  implies  $\int_0^t a_i(s)ds \in I$ ,  $i = 1, 2$ .

As in Section 4 we construct an “infinite dimensional”  $T$ -translation operator associated to 7.1 for  $\lambda = 0$ . Namely, we let  $Q_T^{a_1, a_2}$  be the map that to any  $\varphi \in \tilde{N}$  associates the map  $\theta \mapsto \zeta(T + \theta, \varphi(0))$ , whenever it makes sense to do so. Here  $\zeta(\cdot, p)$  denotes the unique solution of (7.3a) with  $\xi_0 = \varphi(0)$ .

Let  $\Phi: N \rightarrow \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} = \mathbb{R}^{n_1+n_2}$  be the tangent vector field on  $N$  given by  $\Phi(p_1, p_2) = (\Phi_1(p_1), \Phi_2(p_2))$ . The following result similar to Theorem 4.2 holds:

**Proposition 7.1.** *Let  $a_1, a_2, \Phi_1, \Phi_2, N_1, N_2, N, T$  and  $Q^{a_1, a_2}$  be as above. Take  $W \subseteq \tilde{N}$  open and such that the fixed point index of  $Q^{a_1, a_2}$  is defined in  $W$ , then so is  $\deg(\Phi, W_\#)$  and*

$$(7.4) \quad \text{ind}(Q^{a_1, a_2}, W) = (\text{sign } \phi_1)^{\dim N_1} (\text{sign } \phi_2)^{\dim N_2} \deg(\Phi, W_\#).$$

*Sketch of the proof.* The assertion can be proved by following closely the argument of Theorem 4.2 and taking into account the following well-known and easily verified fact of degree theory:

For any given pair of constants  $c_1, c_2 \in \mathbb{R} \setminus \{0\}$  and tangent vector fields  $v_1: N_1 \rightarrow \mathbb{R}^{n_1}$  and  $v_2: N_2 \rightarrow \mathbb{R}^{n_2}$ , admissible on an open  $\mathcal{V} \subseteq N$ , one has

$$\deg(v_c, \mathcal{V}) = (-\text{sign } c_1)^{\dim N_1} (-\text{sign } c_2)^{\dim N_2} \deg(v, \mathcal{V}).$$

where  $v, v_c: N \rightarrow \mathbb{R}^{n_1+n_2}$  are the tangent vector fields on  $N$  given by  $v(p_1, p_2) = (v_1(p_1), v_2(p_2))$  and  $v_c(p_1, p_2) = (c_1 v_1(p_1), c_2 v_2(p_2))$  for all  $(p_1, p_2) \in N$ .  $\square$

Any  $(\lambda, \zeta) \in [0, \infty) \times C_T(N)$ , with  $\zeta$  solution of (7.1), is a  $T$ -periodic pair. Such a pair is *trivial* if  $\lambda = 0$  and  $\zeta$  is constant. An argument that follows very closely the one of Theorem 6.1 yields the following result:

**Proposition 7.2.** *For  $i = 1, 2$ , let  $\Phi_i: N_i \rightarrow \mathbb{R}^{n_i}$  be (continuous) tangent vector fields, and let  $\Xi_i: \mathbb{R} \times N \times N \rightarrow \mathbb{R}^{n_i}$  be tangent to  $N_i$  in the second variable; assume that the  $\Xi_i$ ’s as well as the maps  $a_i: \mathbb{R} \rightarrow \mathbb{R}$ , are continuous,  $T$ -periodic in  $t$ . Suppose also that, for  $i = 1, 2$ , the average of  $\phi_i$  of  $a_i$  is nonzero. Let  $\Omega \subseteq [0, \infty) \times C_T(N)$  be open, and assume that  $\deg(\Phi, \Omega \cap \tilde{N})$  is defined and nonzero. Then  $\Omega$*



contains a connected set of nontrivial  $T$ -periodic pairs of (7.1) whose closure in  $\Omega$  meets the set  $\{(0, \bar{p}) \in \Omega : \Phi(p) = (0)\}$  and is not compact.

**Example 7.3.** Let  $T = 2\pi$  and consider the following system of equations in  $\mathbb{R}^3$

$$\begin{cases} \dot{x}_2 = (2 + \sin(t))(x_2 + x_3), \\ \dot{x}_1 - \dot{x}_3 = |\cos(t)|x_1 - (2 + \sin(t))x_3, \\ -\dot{x}_2 + \dot{x}_3 = -(2 + \sin(t))x_2, \end{cases}$$

that we write more compactly as follows:

$$(7.5) \quad E\dot{x} = A(t)x,$$

where, for  $t \in \mathbb{R}$ ,

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix} \quad \text{and} \quad A(t) = \begin{pmatrix} 0 & 2 + \sin(t) & 2 + \sin(t) \\ |\cos(t)| & 0 & -2 - \sin(t) \\ 0 & -2 - \sin(t) & 0 \end{pmatrix}.$$

Let us now consider the following  $2\pi$ -periodic perturbation of (7.5):

$$(7.6) \quad E\dot{x}(t) = A(t)x(t) + \lambda \mathcal{H}(t, x(t), x(t-r)), \quad \lambda \geq 0,$$

where  $r > 0$  is a given time lag and  $\mathcal{H}: \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is a continuous map which is  $2\pi$ -periodic in its first variable. Multiplying (7.6) on the left by  $E^{-1}$  and setting, for all  $(t, p, q) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$ ,

$$H(t, p, q) = E^{-1}\mathcal{H}(t, p, q), \quad \text{and} \quad B(t) = E^{-1}A(t)$$

we see that (7.6) becomes

$$(7.7) \quad \dot{x}(t) = B(t)x(t) + \lambda H(t, x(t), x(t-r)).$$

Clearly,

$$B(t) = \begin{pmatrix} |\cos(t)| & 0 & 0 \\ 0 & 2 + \sin(t) & 0 \\ 0 & 0 & 2 + \sin(t) \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence, identifying  $\mathbb{R}^3$  with the product space  $\mathbb{R} \times \mathbb{R}^2$ , we see that (7.7) falls into the family of weakly coupled systems (7.1) with  $N_1 = \mathbb{R}$  and  $N_2 = \mathbb{R}^2$ . Let  $\Phi: \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be given by  $\Phi(p; q_1, q_2) := (p, q_1 + q_2, q_1)$ , and take  $\Omega = [0, \infty) \times C_T(\mathbb{R}^3)$ . Since,  $\Phi$  is admissible for the degree in  $\Omega \cap \mathbb{R}^3 = \mathbb{R}^3$  and  $\deg(\Phi, \mathbb{R}^3) = 1$ , by Proposition 7.2, there exists a connected set of nontrivial  $T$ -periodic pairs of (7.7) whose closure meets the set  $\Phi^{-1}(0) = \{(0; 0, 0)\}$  and is not compact. Since solutions of (7.7) are solutions of (7.6) and vice versa, this statement concerns, in fact, the  $T$ -periodic solutions of (7.6).

## 7.2. Periodic perturbations of separated variables Differential-Algebraic

**Equations.** Here, as in the Introduction,  $U \subseteq \mathbb{R}^k \times \mathbb{R}^s$  is open and connected and  $g: U \rightarrow \mathbb{R}^s$  is  $C^\infty$  with the property that  $\partial_2 g(x, y)$  is nonsingular for any  $(x, y) \in U$ . In this way,  $M := g^{-1}(0)$  is a  $C^\infty$  submanifold of  $\mathbb{R}^k \times \mathbb{R}^s$ . We also require that  $f$  and  $h$ , as in Equation (1.4), are continuous and that  $a$  and  $h$  are  $T$ -periodic in  $t$  with the average of  $a$  different from zero.

In what follows, we say that  $(\lambda, (x, y)) \in [0, \infty) \times C_T(U)$  is a  $T$ -periodic pair of (1.4), if  $(x, y)$  is a  $T$ -periodic solution of (1.4) corresponding to  $\lambda$ . According to the convention introduced in (2.1)-(2.2), any  $(p, q) \in U$  is identified with the element  $(\bar{p}, \bar{q})$  of  $C_T(U)$  that is constantly equal to  $(p, q)$ . A  $T$ -periodic pair of the form



$(0, (\bar{p}, \bar{q}))$  will be called *trivial*. This subsection is devoted to the study of the set of  $T$ -periodic pairs of equation (1.4).

Thanks to our assumption on  $g$  it is possible to associate tangent vector fields on  $M$  to the functions  $f$  and  $h$  in (1.4). Consider first maps  $\Psi: U \rightarrow \mathbb{R}^k \times \mathbb{R}^s$  and  $\Upsilon: \mathbb{R} \times U \times U \rightarrow \mathbb{R}^k \times \mathbb{R}^s$  as follows:

$$\begin{aligned}\Psi(p_1, q_1) &= (f(p_1, q_1), -[\partial_2 g(p_1, q_1)]^{-1} \partial_1 g(p_1, q_1) f(p_1, q_1)), \\ \Upsilon(t, (p_1, q_1), (p_2, q_2)) &= \\ &\quad \left( h(t, (p_1, q_1), (p_2, q_2)), -[\partial_2 g(p_1, q_1)]^{-1} \partial_1 g(p_1, q_1) h(t, (p_1, q_1), (p_2, q_2)) \right),\end{aligned}$$

and then define

$$(7.9) \quad \Phi = \Psi|_M \quad \text{and} \quad \Xi = \Upsilon|_{\mathbb{R} \times M \times M}.$$

Since  $T_{(p,q)}M$  coincides with the kernel of the differential of  $g$  at any  $(p, q) \in M$ , it can be easily seen that  $\Phi(p, q) \in T_{(p,q)}M$  and that  $\Xi(t, (p_1, q_1), (p_2, q_2)) \in T_{(p_1, q_1)}M$ , for all  $(t, (p_1, q_1), (p_2, q_2)) \in \mathbb{R} \times M \times M$ . Therefore, the following is a delay differential equation on  $M$ :

$$(7.10) \quad \dot{\zeta}(t) = a(t)\Phi(\zeta(t)) + \lambda\Xi(t, \zeta(t), \zeta(t-r)), \quad \lambda \geq 0.$$

Using Lemma 3.2 it is not difficult to show that (7.10) is equivalent to (1.4), in the sense that  $\zeta = (x, y)$  is a solution of (7.10), on an interval  $I \subseteq \mathbb{R}$ , if and only if so is  $(x, y)$  for (1.4). Thus, we can combine the results of Theorems 3.3 and 6.1. The map  $F: U \rightarrow \mathbb{R}^k \times \mathbb{R}^s$  introduced in (3.7) becomes, in our case,

$$F(x, y) = (f(x, y), g(x, y)).$$

So, with the notation recalled above, the set of trivial  $T$ -periodic pairs can be written as  $\{(0, (\bar{p}, \bar{q})) \in [0, \infty) \times C_T(U) : F(p, q) = (0, 0)\}$ . Also, as in Section 6, given  $\Omega \subseteq [0, \infty) \times C_T(U)$ , we denote by  $\Omega \cap U$  the subset of  $U$  whose points, regarded as constant functions, lie in  $\Omega$ . Namely,  $\Omega \cap U = \{(p, q) \in U : (0, (\bar{p}, \bar{q})) \in \Omega\}$ .

We finally state and prove the following consequence of Theorems 3.3 and 6.1, which is inspired to [3, Th. 5.1].

**Theorem 7.4.** *Let  $U \subseteq \mathbb{R}^k \times \mathbb{R}^s$  be open and connected. Let  $g: U \rightarrow \mathbb{R}^s$ ,  $f: U \rightarrow \mathbb{R}^k$ ,  $a: \mathbb{R} \rightarrow \mathbb{R}$  and  $h: \mathbb{R} \times U \rightarrow \mathbb{R}^k$  be as above. Let also  $F(p, q) = (f(p, q), g(p, q))$ . Given  $\Omega \subseteq [0, \infty) \times C_T(U)$  open, assume that  $\deg(F, \Omega \cap U)$  is well-defined and nonzero. Then, there exists a connected set of nontrivial solution pairs of (1.4) whose closure in  $\Omega$  is noncompact and meets the set  $\{(0, (\bar{p}, \bar{q})) \in \Omega : F(p, q) = (0, 0)\}$  of the trivial  $T$ -periodic pairs of (1.4).*

*Proof.* Let  $\Phi$  and  $\Xi$  be the tangent vector fields defined in (7.9). Let also  $\mathcal{O}$  be the open subset of  $[0, \infty) \times C_T(M)$  given by

$$\mathcal{O} = \Omega \cap ([0, \infty) \times C_T(M)).$$

For any  $Y \subseteq M$ , by  $\mathcal{O} \cap Y$  we mean the set of all those points of  $Y$  that, regarded as constant functions, lie in  $\mathcal{O}$ . Using this convention, one has that  $\Omega \cap Y = \mathcal{O} \cap Y$  and, in particular,  $\Omega \cap M = \mathcal{O} \cap M$ . Thus, Theorem 3.3 implies that

$$|\deg(\Phi, \mathcal{O} \cap M)| = |\deg(\Phi, \Omega \cap M)| = |\deg(F, \Omega \cap U)| \neq 0.$$

Theorem 6.1 yields a connected set  $\Lambda$  of nontrivial  $T$ -periodic pairs of (7.10) whose closure in  $\mathcal{O}$  is not compact and meets the set

$$\{(0, (\bar{p}, \bar{q})) \in \mathcal{O} : \Phi(p, q) = (0, 0)\} = \{(0, \bar{p}, \bar{q}) \in \Omega : F(p, q) = (0, 0)\}.$$

The equivalence of (7.10) with (1.4) imply that each  $(\lambda, (x, y)) \in \Lambda$  is a nontrivial  $T$ -periodic pair of (1.4) as well. Since  $M$  is closed in  $U$ , any relatively closed subset of  $\mathcal{O}$  is relatively closed in  $\Omega$  too and vice versa. Thus, the closure of  $\Lambda$  in  $\mathcal{O}$  coincides with the closure of  $\Lambda$  in  $\Omega$ , and hence  $\Lambda$  fulfills the assertion.  $\square$

#### REFERENCES

- [1] L. Bisconti, *Harmonic solutions to a class of Differential-Algebraic Equations with separated variables*, Electron. J. Differential Equations, Vol. 2012 (2012), No. 2, 1-15.
- [2] F. Brauer and C. Castillo-Chávez, *Mathematical Models in Population Biology and Epidemiology*, Springer-Verlag, Texts in Applied Mathematics Vol. 40, 2000.
- [3] A. Calamai and M. Spadini, *Branches of forced oscillations for a class of constrained ODEs: a topological approach*, NoDEA, online first, DOI: 10.1007/s00030-011-013-1.
- [4] J. Dugundij, *Topology*, Allyn and Bacon series in advanced mathematics, Allyn and Bacon, Boston, 1966.
- [5] M. Farkas, *Dynamical models in biology*, Academic Press, San Diego, USA, 2001.
- [6] M. Furi, M. P. Pera, *A continuation principle for periodic solutions of forced motion equations on manifolds and application to bifurcation theory*, Pacific J. Math. **160** (1993), 219-244.
- [7] M. Furi, M. P. Pera and M. Spadini, *The fixed point index of the Poincaré operator on differentiable manifolds*, Handbook of topological fixed point theory, Brown R. F., Furi M., Góńiewicz L., Jiang B. (Eds.), Springer, 2005.
- [8] M. Furi and M. Spadini, *On the set of harmonic solutions of periodically perturbed autonomous differential equations on manifolds*, Nonlin. Anal., Vol. 29 (1997), No. 8, 963-970.
- [9] M. Furi, M. Spadini, *Periodic perturbations with delay of autonomous differential equations on manifolds*, Adv. Nonlinear Stud. 9, No. 2, 263-276 (2009).
- [10] V. Guillemin and A. Pollack, *Differential-Topology*, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1974.
- [11] Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, Mathematics in Science and Engineering, Vol. 191. Academic Press, Inc., Boston, 1993.
- [12] P. Kunkel and V. Mehrmann, *Differential-Algebraic Equations: Analysis and Numerical Solutions.*, EMS Textbooks in Mathematics, 2006.
- [13] S. Lang, *Differential Manifolds*, Addison Wesley Pubbl. Comp. Inc., Reading, Massachussetts, 1972.
- [14] J. W. Milnor, *Topology from the differentiable viewpoint*, Univ. press of Virginia, Charlottesville, 1965.
- [15] W. M. Oliva, *Functional differential equations on compact manifolds and an approximation theorem*, J. Differential Equations **5** (1969), 483-496.
- [16] M. Spadini, *Harmonic solutions to perturbations of periodic separated variables ODEs on manifolds*, Electron. J. Differential Equations Vol. 2003 (2003), No. 88.